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LETTER TO THE EDITOR

A remark on von Neumann–Wigner type potentials

A A Stahlhofen
Max-Planck-Institut Metallforschung, Institut für Physik, Heisenbergstrasse 1, D-70569 Stuttgart, Germany
and
Max-Planck-Arbeitsgruppe 'Nichtklassische Strahlung', Rudower Chaussee 5, Gebäude 10.16, D-12484 Berlin, Germany

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Abstract. A general formulation of the modulation function approach to von Neumann–Wigner type potentials is given covering recent discussions of such potentials as special cases.

The experimental verification of bound states in the continuum described by a local potential in the one-particle one-channel Schrödinger equation [1] has revived interest in von Neumann–Wigner type potentials [2] with embedded eigenvalues. Recent discussions of such long-ranged oscillating potentials, constructed on trivial as well as nontrivial background, were based on supersymmetric quantum mechanics [3], Darboux transformations [4], the Gel'fand–Levitan formalism [5] or a special modulation function [6].

An extension of the modulation function approach covering the techniques used in the aforementioned approaches is given below; special cases are identified and commented on. (A discussion of examples and suggestions for applications will be presented elsewhere).

Let us consider the radial s-wave Schrödinger equation reading (in simplified units)

$$-\varphi''(r) + V_0(r)\varphi(r) = k^2\varphi(r) \tag{1}$$

with general boundary condition (ρ real)

$$\sin\rho\varphi(0) + \cos\rho\varphi'(0) = 0. \tag{2}$$

Let $\varphi_0(r)$ be a solution of (1) for a given background potential $V_0(r)$ corresponding to $k^2 = k_0^2$ and introduce a new function $\phi(r)$ via

$$\phi(r) := \frac{\varphi_0(r)}{f(r)}. \tag{3}$$

This modulation of $\varphi_0(r)$ by a free function $f(r)$—subject only to the condition of square integrability (and exclusion of singularities in the potential defined below)—defines a solution $\phi(r)$ of the Schrödinger equation (1) with the new potential

$$V(r) = V_0(r) - 2(\ln(f(r)))'' + \frac{f''(r)}{f(r)} - 2\frac{f'(r)\varphi_0'(r)}{f(r)\varphi_0(r)}. \tag{4}$$

The ansatz $\phi(r) = \varphi_0(r)f(r)$ made in [6] is completely equivalent to (3) which is technically more convenient.

Following an idea of [7], the modulating function $f(r)$ is now chosen as

$$f(r) = \left(A + \left(B\int_0^r \varphi_0(y)^2 \mathrm{d}y\right)^m\right)^n \tag{5}$$

where $\varphi_0(r)$ is again a solution of (1) *not* restricted to be an eigenfunction of (1); A, B are free adjustable parameters.

We now focus on the special case $V_0(r) := 0$ with ($k_0^2 = \kappa^2$, $\rho = \pi/2$)

$$\varphi_0(r) = \frac{1}{\kappa}\sin(\kappa r) \tag{6}$$

for simplicity; for a first generalization to angular momentum $l \neq 0$ the corresponding spherical Bessel functions have to be used. With the abreviation $s(r) := (B\int_0^r \varphi_0^2(y)\mathrm{d}y)^m$, the potential (4) resulting from the choice (5) can be written as

$$V(r) = -\frac{nms(r)^m\left((s')^2(r)\,(Am - A) - (s')^2(r)s(r)^m(1+nm)\right)}{s(r)^2\left(A + s(r)^{2m}\right)} - \frac{Anms(r)^{m+1}s''(r) + nms(r)^{2m+1}s''(r)}{s(r)^2\left(A + s(r)^{2m}\right)} - \frac{2nms(r)^m s'(r)}{s(r)\,(A + s(r)^m)}\cot(\kappa r). \tag{7}$$

Depending on the choice of constants A, B and powers n, m it is now straightforward to recover from (3), (5) and (7) previous discussions of von Neumann–Wigner potentials:
(i) For $n := 2, m := 1$ and $B := 4\kappa^3$ with arbitrary $A := a^2$ (and real a) one recognizes immediately the original strategy of [2];
(ii) For $n = m := 1$, $A = 1$ and real B fixed by requiring square integrability of $\phi(r)$ in (3) the results are equivalent to those of the so-called double commutation formalism [8] realized either as Darboux transformations [4], supersymmetric quantum mechanics [3] or as factorization procedure;
(iii) For $n = m := 1$, $B := 1$ and free real A (to be identified with the normalization constant of φ_0) one obtains the ansatz of the Gel'fand–Levitan formalism [5].

The double commutation and the equivalent Gel'fand–Levitan ansatz both reduce (4) to the well known form ($V_0(r) = 0$)

$$V(r) = -2(\ln(f(r)))''. \tag{8}$$

They can be iterated to handle potentials with $n \geqslant 1$ embedded eigenvalues and give the scattering solution as

$$\phi(r,k) = \varphi(r,k) - \frac{\varphi_0(r)\int_0^r \varphi_0(y)\phi(y,k)\mathrm{d}y}{f(r)} \tag{9}$$

where $\varphi(r,k)$ is the scattering solution of (1). These formalisms do not allow us to vary the strength (coupling constant) of the potential once all functions/constants are chosen but contain by construction a free parameter which can be related to the physical parameters of the system.

The choice

$$f(r) = A\exp\left(a\int_0^r \frac{\sin^2\kappa z}{z^\beta}\mathrm{d}z\right) \tag{10}$$

for the modulating function (with constant a and $0 < \beta \leqslant 1$) discussed in [6] overcomes the difficulty of having a fixed coupling constant by the parameter a as a multiplicative factor in the potential. There are two remarks to be made here:
(i) The ansatz (10) originated in fact in [9], and has been employed in potentials with

embedded eigenvalues in [10] (chapter four). All results presented in [6] can be obtained from those in [10] by implementing in [10] the changes

$$\phi_0(x) = \cos\kappa x \to \chi(r) = \frac{1}{\kappa}\sin\kappa r \tag{11}$$

$$g(x)\phi_0(x) = \frac{a\cos^2\kappa x}{x^\beta} \to C(r) = \frac{a\sin^2\kappa r}{r^\beta}$$

due to different boundary conditions in [10] and [6]. The asymptotics of the resulting potentials do not depend on the boundary conditions; as in the general ansatz before, the asymptotics of the eigenfunction is determined by $f(r)$. (The notations of [10] have been used on the left-hand side of (11) and those of [6] on the right-hand side).

(ii) The discussion of [6] is incomplete for $\beta = 1$. For this case, the value of the constant a is subject to a 'resonance condition' [11]: any radial potential with the asymptotics

$$\lim V(r) = \frac{b\sin cr}{r} + O\left(\frac{1}{r^2}\right) \qquad r \to \infty \tag{12}$$

has an embedded eigenvalue at $k^2 = c^2/4$, iff the resonance condition

$$\frac{|b|}{|2c|} > \frac{1}{2} \tag{13}$$

is satisfied. This condition restricts a in [6] to $|a| > 1$.

If a von Neumann–Wigner type potential has an embedded eigenvalue, the value of b has drastic consequences for the scattering problem: for $|b/c| \geqslant 2n$, $n = 1, 2, \ldots$ the scattering is trivial (with a Jost-function $F(k) = 1$ and vanishing phase shift $\delta(k) = 0$); for $|b/c| \neq 2n$, the Jost-function is singular and scattering is no longer trivial.

For $A := 1$ and $n = m := 1$ an analytical continuation of the double commutation (or the Gel'fand–Levitan) formalism defined above across the continuums edge in the negative eigenvalue regime—requiring for $\varphi_0(r)$ a bound state solution in the field of $V_0(r)$—shows the equivalence of this formalism to the equivalence problem of [12]. There, the problem of constructing out of the bound state eigenfunctions of background potentials $V_0(r)$ new potentials having the same eigenvalues and the same phase shift has been solved analytically; equations (3), (4), (8) and (9) given here agree with the corresponding equations given there (upon correction of a sign error in equation (2.12) of [12]).

The ansatz presented here has for $V_0(r) = 0$ and $n, m > 1$ the advantage of being applicable to the one-dimensional Schrödinger equation defined over the axis while approaches like the double commutation or the Gel'fand–Levitan formalisms lead to singular potentials.

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